

GRAPHS ASSOCIATED WITH THE MAP $X \mapsto X + X^{-1}$ IN FINITE FIELDS OF CHARACTERISTIC THREE

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ABSTRACT. In [Ugo11] we described the structure of the graphs associated with the iterations of the map $x \mapsto x + x^{-1}$ over finite fields of characteristic two. In this paper we extend our study to finite fields of characteristics three.

1. INTRODUCTION

Let \mathbf{F}_q be a finite field with q elements for some positive integer q . We can define a map ϑ on $\mathbf{P}^1(\mathbf{F}_q) = \mathbf{F}_q \cup \{\infty\}$ in such a way:

$$\vartheta(x) = \begin{cases} x + x^{-1} & \text{if } x \neq 0, \infty \\ \infty & \text{if } x = 0 \text{ or } \infty \end{cases}$$

We associate a graph with the map ϑ over \mathbf{F}_q , labelling the vertices of the graph by the elements of $\mathbf{P}^1(\mathbf{F}_q)$. Moreover, if $\alpha \in \mathbf{P}^1(\mathbf{F}_q)$ and $\beta = \vartheta(\alpha)$, then a directed edge connects the vertex α with the vertex β . If $\gamma \in \mathbf{P}^1(\mathbf{F}_q)$ and $\vartheta^k(\gamma) = \gamma$, for some positive integer k , then γ belongs to a cycle of length k or a divisor of k . The smallest among these integers k is the period l of γ with respect to the map ϑ and the set $\{\vartheta^i(\gamma) : 0 \leq i < l\}$ is the cycle of length l containing γ . An element γ belonging to a cycle can be the root of a reverse-directed tree, provided that $\gamma = \vartheta(\alpha)$, for some α which is not contained in any cycle.

In [Ugo11] we dealt with the characteristic 2 case. There we noticed that the map ϑ is strictly related to the duplication map over Koblitz curves. Later we carried out some experiments in characteristics greater than 5, but the resulting graphs seemed not to present notable symmetries.

In characteristics 3 and 5, in analogy with our previous work [Ugo11], the graphs exhibit remarkable symmetries. In this paper we present the characteristic 3 case.

In characteristic three the structure of the graphs can be described relying upon the fact that ϑ is conjugated to the inverse of the square mapping. Fixed a finite field \mathbf{F}_{3^n} we provide the following information about the graph associated with ϑ :

- the lengths, the number of the cycles and the number of the connected components (Theorem 2.2);
- the depth and the properties of the trees (Theorem 2.4).

2. STRUCTURE OF THE GRAPHS IN CHARACTERISTIC THREE

In characteristic three the iterations of the map ϑ can be studied relying upon the consideration that ϑ is conjugated to the inverse of the square map. Indeed, if x is any element of a field of characteristic 3, then

$$(2.1) \quad \vartheta(x) = \psi \circ s \circ \psi(x),$$

where s and ψ are functions defined on $\mathbf{P}^1(\mathbf{F}_{3^n})$ as follows

$$s(x) = \begin{cases} x^{-2} & \text{if } x \in \mathbf{F}_{3^n}^* \\ 0 & \text{if } x = \infty \\ \infty & \text{if } x = 0 \end{cases} \quad \psi(x) = \begin{cases} \frac{x+1}{x-1} & \text{if } x \in \mathbf{P}^1(\mathbf{F}_{3^n}) \setminus \{1, \infty\} \\ 1 & \text{if } x = \infty \\ \infty & \text{if } x = 1 \end{cases}$$

We note that ψ is a self-inverse map over $\mathbf{P}^1(\mathbf{F}_{3^n})$, namely $\psi^2(x) = x$ for any $x \in \mathbf{P}^1(\mathbf{F}_{3^n})$. Therefore the following holds for the k -th iterate of ϑ :

$$\vartheta^k(x) = \psi \circ s^k \circ \psi(x).$$

We say that an element $x \in \mathbf{P}^1(\mathbf{F}_{3^n})$ is ϑ -periodic (resp. s -periodic) iff $\vartheta^k(x) = x$ (resp. $s^k(x) = x$), for some positive integer k . The smallest such k will be called the period of x with respect to the map ϑ (resp. s).

We prove the following characterization of ϑ -periodic points.

Lemma 2.1. *Let n be a positive integer.*

- *The elements 1 and -1 are ϑ -periodic of period 2 and form a cycle of length 2.*
- *The element ∞ is ϑ -periodic of period 1.*
- *An element $\alpha \in \mathbf{P}^1(\mathbf{F}_{3^n}) \setminus \{-1, 1, \infty\}$ is ϑ -periodic of period k if and only if $\psi(\alpha)$ is s -periodic of period k . Moreover, the integer k is odd and is equal to the multiplicative order $\text{ord}_d(-2)$ of -2 in $(\mathbf{Z}/d\mathbf{Z})^*$, where d is the multiplicative order of $\psi(\alpha)$ in $\mathbf{F}_{3^n}^*$.*

Proof. An element $\alpha \in \mathbf{P}^1(\mathbf{F}_{3^n})$ is ϑ -periodic if and only if there exists a positive integer k such that $\vartheta^k(\alpha) = \psi \circ s^k \circ \psi(\alpha) = \alpha$. Let $\beta = \psi(\alpha)$. We have the following equivalences:

$$\vartheta^k(\alpha) = \alpha \Leftrightarrow \psi \circ s^k \circ \psi(\alpha) = \alpha \Leftrightarrow s^k(\psi(\alpha)) = \psi(\alpha).$$

In virtue of what we have just proved α is ϑ -periodic of period k if and only if $\beta = \psi(\alpha)$ is s -periodic of period k .

If $\alpha \in \mathbf{P}^1(\mathbf{F}_{3^n}) \setminus \{-1, 1, \infty\}$, then $\psi(\alpha) \notin \{0, \infty\}$. Therefore, α is ϑ -periodic of period k if and only if $\beta^{(-2)^k} = \beta$, namely $\beta^{(-2)^k - 1} = 1$. This latter is true if and only if d divides $(-2)^k - 1$. That means that $k = \text{ord}_d(-2)$. Since $(-2)^k - 1$ is an odd integer, this is possible iff the multiplicative order of β in $\mathbf{F}_{3^n}^*$ is odd.

Finally, consider the elements $1, -1, \infty$. Since $\vartheta(-1) = 1$, $\vartheta(1) = -1$ and $\vartheta(\infty) = \infty$ the first two statements of the claim are proved. \square

In the following Theorem the lengths and the number of cycles of the graph associated with ϑ over \mathbf{F}_{3^n} are given.

Theorem 2.2. *Let n be a positive integer and $D = \{d_1, \dots, d_m\}$ the set of the distinct odd integers greater than 1 which divide $3^n - 1$. Denote by $\text{ord}_{d_i}(-2)$ the multiplicative order of -2 in $(\mathbf{Z}/d_i\mathbf{Z})^*$. Consider the set*

$$L = \{\text{ord}_{d_i}(-2) : 1 \leq i \leq m\} = \{l_1, \dots, l_r\}$$

of cardinality r , where $r \leq m$, and the map

$$\begin{aligned} l : D &\rightarrow L \\ d_i &\mapsto \text{ord}_{d_i}(-2). \end{aligned}$$

Then:

- $L \cap \{1, 2\} = \emptyset$;
- *the length of a cycle in the graph associated with ϑ over $\mathbf{P}^1(\mathbf{F}_{3^n})$ is a positive integer belonging to $L \cup \{1, 2\}$;*
- *there is one cycle of length 2 formed by 1 and -1 and one cycle of length 1 formed by ∞ ;*
- *for any $1 \leq k \leq r$ there are*

$$c_k = \frac{1}{l_k} \cdot \sum_{d_i \in l^{-1}(l_k)} \varphi(d_i)$$

cycles of length l_k ;

- *the number of connected components of the graph is*

$$2 + \sum_{k=1}^r c_k.$$

Proof. Since any element of L is equal to $\text{ord}_d(-2)$ for some odd integer $d > 1$, then 1 is not contained in L . Moreover 2 is not contained in L too. In fact, $\text{ord}_d(-2) = 2$ if and only if $d = 3$. But this is not possible, because 3 does not divide $3^n - 1$.

In Lemma 2.1 we proved that ± 1 are the only ϑ -periodic elements of order 2, while ∞ is ϑ -periodic of period 1. All ϑ -periodic elements of $\mathbf{P}^1(\mathbf{F}_{3^n}) \setminus \{-1, 0, 1\}$ have odd period k , where k is the multiplicative order of -2 in $(\mathbf{Z}/d\mathbf{Z})^*$, for some odd integer d which divides $3^n - 1$. Therefore, the length of a cycle is an integer belonging to $L \cup \{1, 2\}$.

Take an odd divisor $d_i > 1$ of $3^n - 1$. In $\mathbf{F}_{3^n}^*$ there are $\varphi(d_i)$ elements of order d_i . Since ψ is a bijection on $\mathbf{P}^1(\mathbf{F}_{3^n})$, then each of these elements is of the form $\psi(\alpha)$ for some $\alpha \in \mathbf{P}^1(\mathbf{F}_{3^n})$.

Consider an element $l_k \in L$. Since $\text{ord}_{d_i}(-2) = l_k$ if and only if $d_i \in l^{-1}(l_k)$, then the number of cycles of length l_k is given by c_k . Moreover, since any element of $\mathbf{P}^1(\mathbf{F}_{3^n})$ is finally periodic, we conclude that the number of connected components of the graph is equal to the number of the cycles. \square

We aim at describing the trees rooted at periodic elements of $\mathbf{P}^1(\mathbf{F}_{3^n})$. Before proceeding, we note that the elements 1 and -1 are not roots of any tree. This can be easily seen, since $\vartheta(x) = \pm 1$ iff $x^2 \pm x + 1 = 0$. These two equations are equivalent to the equations $(x - 1)^2 = 0$ and $(x + 1)^2 = 0$, since the underlying field has characteristic 3, and the roots of the two equations are respectively 1 and -1 .

Before studying the structure of the trees, we prove the following preliminary result.

Lemma 2.3. *Let n be a positive integer and 2^e , for some positive integer e , the greatest power of 2 dividing $3^n - 1$. Let $\gamma \in \mathbf{F}_{3^n}$ be a non- ϑ -periodic point (in particular $\gamma \notin \{1, -1\}$). Then, $\vartheta(x) = \gamma$ for exactly two distinct elements $x \in \mathbf{F}_{3^n}$, provided that $\text{ord}(\psi(\gamma)) \not\equiv 0 \pmod{2^e}$, where $\text{ord}(\psi(\gamma))$ is the multiplicative order of $\psi(\gamma)$ in $\mathbf{F}_{3^n}^*$. If, on the contrary, $\text{ord}(\psi(\gamma)) \equiv 0 \pmod{2^e}$, then there is no $x \in \mathbf{F}_{3^n}$ such that $\vartheta(x) = \gamma$.*

Proof. Take γ as in the hypotheses. We note that, if $\vartheta(x) = \gamma$, then $x \notin \{-1, 0, 1\}$, since $\vartheta(-1) = 1, \vartheta(1) = -1$ and $\vartheta(0) = \infty$, but $\gamma \in \mathbf{F}_{3^n} \setminus \{1, -1\}$. Hence, there exists $x \in \mathbf{F}_{3^n}$ such that $\vartheta(x) = \gamma$ iff $\psi \circ s \circ \psi(x) = \gamma$, namely iff $\psi(x)^{-2} = \psi(\gamma)$. This is equivalent to saying that $\psi(\gamma)$ is a quadratic residue in \mathbf{F}_{3^n} . This is true iff $\psi(\gamma)^{(3^n-1)/2} = 1$ in $\mathbf{F}_{3^n}^*$, namely iff $\text{ord}(\psi(\gamma)) \mid \frac{3^n-1}{2}$. This latter is equivalent to saying that $\text{ord}(\psi(\gamma)) \not\equiv 0 \pmod{2^e}$. \square

In the following result the depth of the reversed binary trees rooted at ϑ -periodic elements is given.

Theorem 2.4. *Let $\alpha \in \mathbf{P}^1(\mathbf{F}_{3^n}) \setminus \{1, -1\}$ be a ϑ -periodic point. Then, α is the root of a reversed binary tree of depth e , where 2^e is the greatest power of 2 which divides $3^n - 1$. In particular:*

- *there are 2^{k-1} vertices at any level $1 \leq k \leq e$;*
- *the root has one child and all the other vertices at any level $k < e$ have two children;*
- *if $\beta \in \mathbf{F}_{3^n}$ belongs to the level $k > 0$ of the tree rooted at α , then 2^k is the greatest power of 2 dividing $\text{ord}(\psi(\beta))$.*

Proof. If $\alpha = \infty$, then α is ϑ -periodic of period 1. Indeed, $\vartheta(\infty) = \infty$. Moreover $\vartheta(x) = \infty$ iff $x = \infty$ or 0. The point 0 is the only vertex belonging to the first level of the tree rooted at ∞ . Moreover, $\psi(0) = -1$, which has order 2 in $\mathbf{F}_{3^n}^*$.

If $\alpha \in \mathbf{F}_{3^n} \setminus \{-1, 1\}$ is a ϑ -periodic element, then $\psi(\alpha) \in \mathbf{F}_{3^n}^*$ and finding all the elements β such that $\vartheta(\beta) = \alpha$ amounts to finding all the elements β such that $\psi \circ s \circ \psi(\beta) = \alpha$. This latter is equivalent to $s(\psi(\beta)) = \psi(\alpha)$, namely $\psi(\beta)^2 = \psi(\alpha)^{-1}$. According to Lemma 2.1 the order of $\psi(\alpha)$, and consequently $\psi(\alpha)^{-1}$, is odd. Hence, $(\psi(\alpha)^{-1})^{(3^n-1)/2} = 1$ in \mathbf{F}_{3^n} . Therefore, $\psi(\alpha)^{-1}$ is a quadratic residue in

$\mathbf{F}_{3^n}^*$ and there are two distinct roots $r_1, r_2 = -r_1$ of $x^2 - \psi(\alpha)^{-1}$ in \mathbf{F}_{3^n} . Being the map ψ a bijection on $\mathbf{P}^1(\mathbf{F}_{3^n})$, it follows that $r_1 = \psi(\beta_1)$ and $r_2 = \psi(\beta_2)$ for two distinct elements β_1 and β_2 in \mathbf{F}_{3^n} . Moreover, since α is ϑ -periodic, one among β_1 and β_2 , let us say β_1 , is ϑ -periodic too and consequently r_1 has odd order. On the contrary β_2 is not ϑ -periodic and $\text{ord}(\psi(\beta_2)) = 2 \cdot \text{ord}(\psi(\beta_1))$, namely the highest of 2 which divides $\text{ord}(\psi(\beta_2))$ is 2.

The remaining statements regarding the levels $k > 0$ will be proved by induction on k . Let us first consider the level $k = 1$. If $e = 1$, then there are no elements at the second level of the tree by Lemma 2.3. In the case $e > 1$ consider the only element γ belonging to the first level of the tree rooted at some ϑ -periodic point of $\mathbf{P}^1(\mathbf{F}_{3^n}) \setminus \{-1, 1\}$. We have proved that 2 is the greatest power of 2 which divides $\text{ord}(\psi(\gamma))$. In virtue of Lemma 2.3 there are exactly two elements belonging to the level 2 of the tree, whose image under the action of the map ϑ is γ .

Now we proceed with the inductive step. Suppose that for some integer $k > 1$ such that $k - 1 < e$ there are 2^{k-2} elements at the level $k - 1$ of the tree and that each of these elements has two children. Moreover, if γ is one of the elements at the level $k - 1$, then 2^{k-1} is the greatest power of 2 which divides $\text{ord}(\psi(\gamma))$. Let β any of the children of γ . Since $\vartheta(\beta) = \gamma$, we have that $\psi(\beta)^{-2} = \psi(\gamma)$. Then 2^k is the highest power of 2 which divides $\text{ord}(\psi(\beta))$. Finally, if $k < e$, then β has two children, while, if $k = e$, then β has no child by Lemma 2.3. \square

2.1. An example: the graph associated with ϑ over the field \mathbf{F}_{3^3} . The field with 27 elements can be constructed as the splitting field over \mathbf{F}_3 of the Conway polynomial $x^3 - x + 1$. In particular, if α denotes a root of such a polynomial, $\mathbf{P}^1(\mathbf{F}_{3^3}) = \{\alpha^i : 0 \leq i \leq 25\} \cup \{0\} \cup \{\infty\}$.

Below are represented the 3 connected components of the graph. The labels of the vertices are the exponents of the powers α^i , for $0 \leq i \leq 25$, the zero element (denoted by '0') and the point ∞ .

We notice that, according with Theorem 2.2, the elements $\alpha^0 = 1$ and $\alpha^{13} = -1$ form a cycle of length 2, while the cycle formed by ∞ has length 1. Moreover, the set of odd integer divisors of $3^3 - 1$ greater than 1 is $D = \{13\}$. Since $\text{ord}_{13}(-2) = 12$, then there is $\frac{1}{13} \cdot \varphi(13) = 1$ cycle of length 12.

Finally, in accordance with Theorem 2.4, any element belonging to a cycle is root of a binary tree of depth 1.



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